Available online at www.sciencedirect.com
SCIENCE DIRECT®
LINEAR ALGEBRA AND ITS APPLICATIONS
ELSEVIER
Linear Algebra and its Applications 364 (2003) 105–124
www.elsevier.com/locate/laa

Positivity of principal minors, sign symmetry and stability

Daniel Hershkowitz*, Nathan Keller

Department of Mathematics, Technion—Israel Institute of Technology, Haifa 32000, Israel

Received 5 May 2002; accepted 13 August 2002

Submitted by H. Schneider

Abstract

The relation between positivity of principal minors, sign symmetry and stability of matrices is studied. It is shown that for sign symmetric matrices, having positive principal minors is equivalent to stability, to D-stability, and to having a positive scaling into a stable matrix. The relation between spectra of matrices some of whose powers have positive principal minors and matrices whose corresponding powers have positive sums of principal minors of each order is studied as well. It is shown that for matrices of order less than 4 these two classes share the same spectra. The relation of these classes and stability is studied, in particular for sign symmetric matrices and for anti-sign symmetric matrices.

Keywords: P-matrix; Q-matrix; Sign symmetric matrix; Weakly sign symmetric matrix; Stable matrix; Semistable matrix; P-matrix powers; D-stability

1. Introduction

This paper deals with the relation between positive stability, positivity of principal minors and sign symmetry for matrices with real principal minors.

Positive stable [*semistable*] matrices are matrices all of whose eigenvalues lie in the open [closed] right half-plane. Positive stability, as well as other types of stability, play important role in various applications and thus have been intensively investigated in the last two centuries, see, e.g., the survey paper [9]. Further in the paper we use the term "stability" for positive stability.

* Corresponding author. Tel./fax: +972-4829-4282.
E-mail address: hershkow@techunix.technion.ac.il (D. Hershkowitz).

0024-3795/03/$ - see front matter
doi:10.1016/S0024-3795(02)00547-5

P-matrices [*P_0-matrices*] are matrices all of whose principal minors are positive [nonnegative]. P-matrices, first introduced in [5], arose as a common generalization of some known classes of matrices such as nonsingular M-matrices, totally positive matrices and positive definite matrices, and appear in various applications, see e.g., [1] for applications in physics, [16] for applications in economics, etc.

Q-matrices [*Q_0-matrices*] are matrices whose sums of principal minors of the same order are all positive [nonnegative]. Obviously, the property of being a Q-matrix depends only on the spectrum of the matrix. *P_0^+-matrices* are Q-matrices which are also P_0-matrices, that is, matrices all of whose principal minors are nonnegative, with at least one positive principal minor of each order. These matrices appear in the research of various types of stability such as D-stability (see Definition 2.7).

For a finite or infinite set S of positive integers, a square matrix A is said to be a *P^S-matrix* [Q^S-matrix] if A^k is a P-matrix [Q-matrix] for all $k \in S$. Since in our discussion we put some emphasis on $P^{\{1,2\}}$-matrices and $Q^{\{1,2\}}$-matrices, we call these matrices P^2-matrices and Q^2-matrices. A matrix is said to be a PM-matrix [QM-matrix] if *all* its powers are P-matrices [Q-matrices].

For subsets α and β of $\{1, \ldots, n\}$ we denote by $A(\alpha|\beta)$ the submatrix of A with rows indexed by α and columns indexed by β. If $|\alpha| = |\beta|$ then we denote by $A[\alpha, \beta]$ the corresponding minor. We denote $A(\alpha) = A(\alpha|\alpha)$ and $A[\alpha] = A[\alpha|\alpha]$. A matrix A is called *sign symmetric* if $A[\alpha|\beta]A[\beta|\alpha] \geqslant 0$ for all $\alpha, \beta \subset \{1, \ldots, n\}$ such that $|\alpha| = |\beta|$. A matrix A is called *weakly sign symmetric* if $A[\alpha|\beta]A[\beta|\alpha] \geqslant 0$ for all $\alpha, \beta \subset \{1, \ldots, n\}$ such that $|\alpha| = |\beta| = |\alpha \cap \beta| + 1$, that is, if the products of symmetrically located (with respect to the main diagonal) almost principal minors are nonnegative.

The research of the relationship between stability, positivity of principal minors and sign symmetry was motivated by a research problem by Taussky [17] calling for investigation of the common properties of totally positive matrices, nonsingular M-matrices and positive definite matrices. Stability, positivity of principal minors and weak sign symmetry are amongst those common properties. In particular, there has been some focus on the question to what extent positivity of principal minors and (weak) sign symmetry imply stability. In this context, Carlson [3] conjectured that a weakly sign symmetric P-matrix is necessarily stable. The conjecture was disproved by Holtz [12]. Carlson did prove a weaker version of his conjecture, that is

Theorem 1.1. *A sign symmetric P-matrix is stable.*

We remark that sign symmetry is shared by totally positive and positive definite matrices but not by nonsingular M-matrices (see e.g., [8]).

In Section 2 we deal with the inverse direction of Theorem 1.1. Obviously, a stable P-matrix is not necessarily sign symmetric. This can be illustrated by the matrix

$$A = \begin{bmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{bmatrix},$$

where ε is a real number. Thus, the only possible opposite direction of Theorem 1.1 is the question whether a sign symmetric stable matrix is necessarily a P-matrix. In the following section we prove this claim. It thus follows that for the class of sign symmetric matrices, positivity of principal minors and stability are equivalent. Furthermore, we show that these properties are equivalent also to D-stability and to having a positive scaling into a stable matrix. The possibility of generalizing our results to the class of Q-matrices and related questions is discussed in Section 3.

Next we deal with the relation between P^S- and Q^S-matrices and sign symmetry. The proof of Theorem 1.1 uses the fact that for a sign symmetric P-matrix A and a positive diagonal matrix D, the matrix $(DA)^2$ is a P-matrix. That proof does not really need that $(DA)^2$ is a P-matrix for every positive scaling D, but rather the weaker condition that $(DA)^2$ has no nonpositive real eigenvalues. For the latter it is enough to require that $(DA)^2$ is a Q-matrix for every positive scaling D. In Section 4 we study matrices satisfying this and related conditions. We first show that the property that the square of every positive scaling of a matrix is a P_0-matrix characterizes sign symmetric matrices. We then discuss the P-matrix version of this result, and as a consequence we restate Theorem 1.1 to claim that matrices all of whose positive scalings are P^2-matrices are necessarily stable. We then study matrices all of whose positive scalings are Q^2-matrices, and show that anti-sign symmetric P-matrices whose square is a P_0^+-matrix are stable.

These results raise the natural question as to how far is a P-matrix A from being stable. This question is answered by Kellogg [13] in terms of the width of a wedge around the negative x-axis which is free from eigenvalues of A. First, Kellogg proved the following:

Theorem 1.2 [13, Theorem 4]. *A set of numbers is the spectrum of some P-matrix if and only if it is the spectrum of some Q-matrix, that is, if and only if all the elementary functions of these numbers are positive.*

Then he proved

Theorem 1.3 [13, Corollary 1]

(i) *Let A be an $n \times n$ Q_0-matrix. Then all nonzero eigenvalues λ of A satisfy*

$$|\arg(\lambda)| \leqslant \pi - \frac{\pi}{n}. \tag{1.1}$$

(ii) *Let A be an $n \times n$ Q-matrix. Then all eigenvalues λ of A satisfy*

$$|\arg(\lambda)| < \pi - \frac{\pi}{n}. \tag{1.2}$$

Furthermore, the bound $\pi - (\pi/n)$ in (1.1) *and in* (1.2) *is sharp.*

Since, by Theorem 1.3, P-matrices of order greater than 2 are not necessarily stable, it is of interest to check a seemingly more restricted class, that is, matrices some of whose powers are P-matrices. In view of Theorem 1.2 it would be interesting to investigate also the relation between spectral properties of such matrices and matrices whose corresponding powers are Q-matrices. In this context, Hershkowitz and Johnson [10] asked whether PM-matrices and QM-matrices share the same spectra. In Section 5 we discuss the more general version of this question, referring to P^S- and Q^S-matrices, where S is a set of positive integers. In particular, we show that indeed for matrices of order less than 4, P^S- and Q^S-matrices share the same spectra.

Section 6 is devoted to the study of stability of P^2- and Q^2-matrices. While such matrices of order less than 4 are stable, we show that 4×4 Q^2-matrices are not necessarily stable. In fact, we show that for every finite set S of positive integers, 4×4 Q^S-matrices are not necessarily stable. The question concerning P^S-matrices remains open.

2. Sign symmetric P-matrices

The matrices we discuss are all assumed to have real principal minors. In this section we show that for sign symmetric matrices positivity of principal minors and stability are equivalent. In order to prove this result we need a few lemmas.

Lemma 2.1. *Let A be a sign symmetric matrix. Then A^2 is a P_0-matrix. Furthermore, if A has at least one nonzero principal minor of each order then A^2 is a P_0^+-matrix.*

Proof. Let A be a sign symmetric $n \times n$ matrix. By the Cauchy–Binet formula (see e.g., [6, p. 9]) it follows that for every subset α of $\{1, \ldots, n\}$ we have

$$A^2[\alpha] = \sum_{\beta \subset \{1,\ldots,n\},\ |\beta|=|\alpha|} A[\alpha|\beta]A[\beta|\alpha] \geqslant 0 \tag{2.1}$$

proving that A^2 is a P_0-matrix. Furthermore, note that if $A[\alpha] \neq 0$ then we have a strict inequality in (2.1), implying that if A has at least one nonzero principal minor of each order then A^2 is a P_0^+-matrix. □

Lemma 2.2. *Let A be a sign symmetric matrix. Then A has no nonzero eigenvalues on the imaginary axis.*

Proof. Let A be a sign symmetric matrix. By Lemma 2.1, the matrix A^2 is a P_0-matrix and thus, by Theorem 1.3, the matrix A^2 has no negative real eigenvalues. It follows that A has no nonzero eigenvalues on the imaginary axis. □

Notation 2.3. For a square matrix A we denote by $\sigma(A)$ the spectrum of A.

Lemma 2.4. *Let A be a stable sign symmetric $n \times n$ matrix. Then all principal submatrices of A are semistable. Furthermore, at least one principal submatrix of A of each order is stable.*

Proof. We first prove that all principal submatrices of A are semistable. Let α be a subset of $\{1, \ldots, n\}$ of cardinality k and let $\sigma(A(\alpha)) = \{\delta_1, \ldots, \delta_k\}$. Let D be the diagonal matrix $\operatorname{diag}(d_1, \ldots, d_n)$ defined by

$$d_j = \begin{cases} 1, & j \in \alpha, \\ 0, & j \notin \alpha. \end{cases}$$

We have

$$\sigma(DA) = \{\delta_1, \ldots, \delta_k, 0, 0, \ldots, 0\}.$$

Denote

$$D_t = (1-t)I_n + tD, \quad 0 \leqslant 1 \leqslant t.$$

Then $D_0 = I$ and $D_1 = D$, and so $D_0A = A$ and $D_1A = DA$. Note that the matrix D_tA is a sign symmetric matrix (with real principal minors) for all t, $0 \leqslant t \leqslant 1$, and thus, by Lemma 2.2, D_tA has no nonzero eigenvalues on the imaginary axis. The matrix A is stable and thus nonsingular. It follows that for all t, $0 \leqslant t < 1$, the matrix D_tA is nonsingular. Therefore, D_tA has no eigenvalues on the imaginary axis for all t, $0 \leqslant t < 1$. Since $\sigma(D_0A)$ is contained in the open right half-plane, and since $\sigma(D_tA)$ depends continuously on t, it follows that $\sigma(D_1A)$ is contained in the closed right half-plane. Since

$$\sigma(DA) = \{\delta_1, \ldots, \delta_k, 0, 0, \ldots, 0\},$$

it follows that $A(\alpha)$ is semistable. Now, consider all principal submatrices of A of some order k. If all are singular then the sum of the principal minors of order k is 0, contradicting the assumption that A is stable (and hence a Q-matrix). Thus, at least one principal submatrix $A(\alpha)$ of A of order k is nonsingular. Since, as is proven above, $A(\alpha)$ is semistable, it follows by Lemma 2.2 that $A(\alpha)$ is stable. □

Next, we quote a theorem of Koteljanskij [14].

Theorem 2.5. *Let A be a weakly sign symmetric matrix having positive leading principal minors. Then A is a P-matrix.*

We are now able to prove the main theorem of this section.

Theorem 2.6. *Let A be a sign symmetric $n \times n$ matrix. The following are equivalent:*

(i) *The matrix A is stable.*
(ii) *The matrix A has positive leading principal minors.*
(iii) *The matrix A is a P-matrix.*

Proof. (i) $\Rightarrow$ (ii). We prove this implication by induction on n. The case $n = 2$ is trivial. Let A be a sign symmetric stable $n \times n$ matrix. By Lemma 2.4, the matrix A has a principal submatrix of order $n-1$ which is stable. This submatrix is also a sign symmetric matrix and thus, by the induction assumption, it has positive leading principal minors. Without loss of generality we may assume that this submatrix is $A(\{1,\dots,n-1\})$. Since A is stable (with a real characteristic polynomial), we have $\det(A) > 0$, and it thus follows that A has positive leading principal minors.

(ii) $\Rightarrow$ (iii) follows by Theorem 2.5.

(iii) $\Rightarrow$ (i) is given in Theorem 1.1. □

Definition 2.7. A matrix A is D-stable if for every positive diagonal matrix D, the matrix DA is stable.

D-stable matrices appear in various applications such as chemical networks and economics (see e.g., [15]). D-stability has been studied extensively (see [9] for references). D-stability is usually a property which is far stronger than stability. Theorem 2.6 allows us to show that for the class of sign symmetric matrices these properties are equivalent. Furthermore, we show that these properties are equivalent to having a positive scaling into a stable matrix.

Theorem 2.8. *Let A be a sign symmetric matrix. The following are equivalent*:

(i) *The matrix A is stable.*
(ii) *The matrix A has positive leading principal minors.*
(iii) *The matrix A is a P-matrix.*
(iv) *The matrix A is D-stable.*
(v) *There exists a positive diagonal matrix D such that the matrix DA is stable.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is proven in Theorem 2.6.

(iii) $\Rightarrow$ (iv). Let A be a sign symmetric P-matrix, and let D be a positive diagonal matrix. Note that DA too is a sign symmetric P-matrix. Our claim follows by Theorem 1.1.

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (iii). If there exists a positive diagonal matrix D such that the matrix DA is stable then, by Theorem 2.6, the matrix DA is a P-matrix. Since D is a positive diagonal matrix, it follows that the matrix A is a P-matrix. □

We conclude this section with a few open problems, motivated by the discussion.

Problem 2.9. It is easy to see that our method of proving Theorem 2.6 does not work for *weakly* sign symmetric matrices, since it is based on the fact that A^2 is a P-matrix, which does not necessarily hold for weakly sign symmetric matrices. Therefore, we ask:

Let A be a weakly sign symmetric stable matrix. Is A necessarily a P-matrix?

This question can be asked both in the general case, in which Carlson's conjecture does not hold [12] and in the case of matrices of order 4, for which Carlson's conjecture holds [8].

Problem 2.10. One can also ask whether our results can be generalized to P_0-matrices and semistability. While it is easy to show that sign symmetric 2×2 matrices are semistable if and only if they are P_0-matrices, the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

shows that matrices of order $n \geqslant 3$ which are sign symmetric P_0-matrices are not necessarily semistable. Therefore, Theorem 1.1 cannot be generalized to P_0-matrices of order $n \geqslant 3$. One can ask whether Theorem 2.6 can be generalized, that is:

Let A be a sign symmetric semistable matrix. Is A necessarily a P_0-matrix?

In the case of matrices of order 2 the claim does hold. To see this let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a sign symmetric matrix. It is easy to check that the eigenvalues of A are real numbers. If A is semistable then either A is stable, in which case by Theorem 2.6 A is a P-matrix, or 0 is an eigenvalue of A. In the latter case we have $\det(A) = 0$ and thus

$$ad = bc. \tag{2.2}$$

Since A is semistable it is a Q_0-matrix, and hence

$$\operatorname{trace}(A) = a + d \geqslant 0. \tag{2.3}$$

Assume that A is not a P_0-matrix. Then either $a < 0$ or $d < 0$. In either case, it follows from (2.3) that a and d are nonzero numbers having opposite signs. It follows that $ad < 0$, and by (2.2) we have $bc < 0$, in contradiction to the sign symmetry of A. Therefore, our assumption that A is not a P_0-matrix is false.

3. Sign symmetric Q-matrices

In view of Theorem 1.2, it is only natural to try to generalize the results of the previous section by replacing the property of having all principal minors positive by the weaker property of having positive sums of principal minors of the same order. For 3×3 matrices we have Theorem 3.3 below, which is a stronger version of Theorem 2.6 . In order to prove it we first prove

Proposition 3.1. *Let A be a 3×3 Q-matrix such that A^2 is a Q_0-matrix. Then A is stable.*

Proof. Let λ be an eigenvalue of A. Since A is a 3×3 Q-matrix, it follows, by Theorem 1.3, that

$$\lambda \neq 0 \quad \text{and} \quad |\arg(\lambda)| < \frac{2\pi}{3}. \tag{3.1}$$

Since A^2 is a Q_0-matrix it follows, by Theorem 1.3 applied to A^2, that $|\arg(\lambda^2)| \leqslant 2\pi/3$, implying that

$$|\arg(\lambda)| \leqslant \frac{\pi}{3} \quad \text{or} \quad |\arg(\lambda)| \geqslant \frac{2\pi}{3}. \tag{3.2}$$

It follows from (3.1) and (3.2) that $|\arg(\lambda)| \leqslant \pi/3$ and hence A is stable. □

Corollary 3.2. *Let A be a sign symmetric 3×3 Q-matrix. Then A is stable.*

Proof. Since A is sign symmetric it follows, by Lemma 2.1, that A^2 is a P_0-matrix. Our claim now follows from Proposition 3.1. □

Theorem 3.3. *Let A be a 3×3 sign symmetric matrix. The following are equivalent*:

(i) *The matrix A is a P-matrix.*
(ii) *The matrix A is stable.*
(iii) *The matrix A is a Q-matrix.*

Proof. (i) $\Leftrightarrow$ (ii) is proven in Theorem 2.6.
(i) $\Rightarrow$ (iii) is trivial.
(iii) $\Rightarrow$ (ii) is proven in Corollary 3.2. □

Theorem 3.3 does not hold in general for matrices of any order n. In fact, even if we replace the condition that the matrix is a Q-matrix by the stronger condition that the matrix is a P_0^+-matrix, we do not necessarily get stability, as is demonstrated by the following example.

Example 3.4. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The matrix A is a sign symmetric P_0^+-matrix. However, we have $\sigma(A) = \{e^{\pm(2\pi/3)i}, 1, 1, 1\}$, and so A is not stable.

Remark 3.5. Observe that all powers of the matrix A in Example 3.4 are sign symmetric P_0^+-matrices. Thus, that example establishes the claim that there exists an unstable matrix all of whose powers are sign symmetric P_0^+-matrices.

Problem 3.6. While Theorem 3.3 asserts that sign symmetric 3×3 Q-matrices are stable, Example 3.4 demonstrates that for $n > 4$ sign symmetric $n \times nQ$-matrices are not necessarily stable. We have no such example of 4×4 matrices, and so we pose the following two problem:

Let A be a 4×4 *sign symmetric Q-matrix. Is A necessarily stable*?
Let A be a 4×4 *weakly sign symmetric Q-matrix. Is A necessarily stable*?

Ballantine [2] proved that for every matrix A with positive leading principal minors, there exists a positive diagonal matrix D such that AD (or, equivalently, DA) is stable. It is natural to ask whether we can replace the requirement that A has positive leading principal minors by the requirement that A is a P_0^+-matrix. Another motivation for this question is a theorem due to Cross [4], stating that a D-stable matrix is necessarily a P_0^+-matrix. In some sense, the opposite direction of this question is the question whether for every P_0^+-matrix there exists a positive diagonal matrix D such that DA is a stable matrix. Our results allow us to answer both questions in the negative. The matrix A in Example 3.4 is a sign symmetric P_0^+-matrix which is not stable. By Theorem 2.8, it follows that there exists no positive diagonal matrix D such that DA is a stable matrix.

4. Matrices whose scalings have P_0-matrix squares

The proof in [3] of Theorem 1.1 uses the fact that for a sign symmetric P-matrix A and a positive diagonal matrix D, the matrix $(AD)^2$ (or, equivalently, $(DA)^2$) is a P-matrix. That proof does not really need that $(DA)^2$ is a P-matrix for every positive scaling D. It uses only the weaker condition that $(DA)^2$ has no nonpositive real eigenvalues. For the latter it is enough to require that $(DA)^2$ is a Q-matrix for every positive scaling D. In this section we study matrices satisfying this and related conditions. We first show that the property that the square of every positive scaling of a matrix is a P_0-matrix characterizes sign symmetric matrices.

Theorem 4.1. *Let A be a square matrix. The following are equivalent*:

(i) *The matrix A is sign symmetric.*
(ii) *For every positive diagonal matrix D the matrix* $(DA)^2$ *is a* P_0*-matrix.*

Proof. (i) $\Rightarrow$ (ii). Let A be a sign symmetric $n \times n$ matrix and let D be a positive diagonal $n \times n$ matrix. Note that DA is sign symmetric as well. By the Cauchy–Binet formula (see e.g., [6, p. 9]) it follows that for every subset α of $\{1, \ldots, n\}$ we have

$$(DA)^2[\alpha] = \sum_{\beta \subset \{1,\ldots,n\},\ |\beta|=|\alpha|} (DA)[\alpha|\beta](DA)[\beta|\alpha] \geqslant 0. \tag{4.1}$$

(ii) $\Rightarrow$ (i). Let A be a matrix which is not sign symmetric. Then there exist subsets α_0 and β_0 of $\{1, \ldots, n\}$, $|\alpha_0| = |\beta_0|$, such that

$$A[\alpha_0|\beta_0]A[\beta_0|\alpha_0] < 0.$$

For $\varepsilon > 0$ let D_ε be the positive diagonal matrix defined by

$$(D_\varepsilon)_{jj} = \begin{cases} 1, & j \in \alpha_0, \\ \varepsilon, & j \notin \alpha_0. \end{cases}$$

By the Cauchy–Binet formula we have

$$\begin{aligned}(D_\varepsilon A)^2[\beta_0] &= \sum_{\alpha \subset \{1,\ldots,n\},\ |\alpha|=|\beta_0|} (D_\varepsilon A)[\beta_0|\alpha](D_\varepsilon A)[\alpha|\beta_0] \\ &= \sum_{\alpha \subset \{1,\ldots,n\},\ |\alpha|=|\beta_0|} \varepsilon^{|\beta_0 \backslash \alpha_0| + |\alpha \backslash \alpha_0|} A[\beta_0|\alpha]A[\alpha|\beta_0]. \end{aligned} \tag{4.2}$$

Note that the coefficient of the lowest power $\varepsilon^{|\beta_0 \backslash \alpha_0|}$ of ε in (4.2) is the negative product $A[\alpha_0|\beta_0]A[\beta_0|\alpha_0]$. It thus follows that for ε sufficiently small we have $(D_\varepsilon A)^2[\beta_0] < 0$, and so for the positive diagonal matrix D_ε the matrix $(D_\varepsilon A)^2$ is not a P_0-matrix. □

The P-matrix version of Theorem 4.1 is the following:

Theorem 4.2. *Let A be a square matrix with all principal minors nonzero. The following are equivalent*:

(i) *The matrix A is sign symmetric.*
(ii) *For every positive diagonal matrix D the matrix $(DA)^2$ is a P-matrix.*

Proof. (i) $\Rightarrow$ (ii). This implication follows exactly as the corresponding implication in Theorem 4.1, noting that the right-hand side of (4.1) includes the *positive* summand $((DA)[\alpha])^2$.

(ii) $\Rightarrow$ (i) is proven in Theorem 4.1. □

Since a matrix is a P-matrix if and only if every positive scaling of it is a P-matrix, in view of Theorem 4.2, Theorem 1.1 can be restated as

Theorem 4.3. *Let A be a square matrix such that all positive scalings of A are P^2-matrices. Then A is stable.*

We now study the property that the square of every positive scaling of a matrix is a Q-matrix. We start with a necessary condition.

Proposition 4.4. *Let A be a square matrix. If for every positive diagonal matrix D the matrix $(DA)^2$ is a Q-matrix then A^2 is a P_0^+-matrix.*

Proof. Let A be an $n \times n$ matrix. Since A^2 is a Q-matrix it follows that A is nonsingular and so $\det(A^2) > 0$. Assume that A^2 is not a P_0^+-matrix. Since A^2 is a Q-matrix, it follows that for some proper subset α of $\{1, \ldots, n\}$ we have $A^2[\alpha] < 0$. Let $|\alpha| = k$. Similarly to the proof of Theorem 4.1, for $\varepsilon \geqslant 0$ let D_ε be the nonnegative diagonal matrix defined by

$$(D_\varepsilon)_{jj} = \begin{cases} 1, & j \in \alpha, \\ \varepsilon, & j \notin \alpha. \end{cases}$$

It is easy to verify that the only nonzero minor of $(D_0A)^2$ of order k is the negative minor $(D_0A)^2[\alpha] = A^2[\alpha]$. Therefore, $(D_0A)^2$ has a negative sum of principal minors of order k, and by continuity, for positive ε sufficiently small the matrix $(D_\varepsilon A)^2$ has a negative sum of principal minors of order k, in contradiction to the proposition's condition. Therefore, our assumption that A^2 is not a P_0^+-matrix is false. $\square$

The converse of Proposition 4.4 does not hold in general for matrices of order greater than 2, even if we replace the requirement that A^2 is a P_0^+-matrix by the stronger requirement that A is a P^2-matrix, as is demonstrated by the following example.

Example 4.5. The matrix

$$A = \begin{bmatrix} 9 & -100 & 10 \\ 1 & 10 & 10 \\ 10 & 10 & 300 \end{bmatrix}$$

is a P^2-matrix. Nevertheless, for the positive diagonal matrix $D = \operatorname{diag}\{1, 1, 0.01\}$ we have

$$(DA)^2 = \begin{bmatrix} -18 & -1899 & -880 \\ 20 & 1 & 140 \\ 1.3 & -8.7 & 11 \end{bmatrix}.$$

Note that $(DA)^2$ is not a Q-matrix since it has a negative trace.

The converse of Proposition 4.4 does hold for 2×2 matrices.

Proposition 4.6. *Let A be a 2×2 matrix. The following are equivalent*:

(i) *For every positive diagonal matrix D the matrix $(DA)^2$ is a Q-matrix.*
(ii) *The matrix A^2 is a P_0^+-matrix.*

Proof. (i) $\Rightarrow$ (ii) is proven in Proposition 4.4.

(ii) $\Rightarrow$ (i). Let

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$$

be such that A^2 is a P_0^+-matrix and let $D=\operatorname{diag}\{d_1,d_2\}$ be a positive diagonal matrix. Since

$$\det((DA)^2)=\det(D)^2\det(A)^2>0,$$

all we have to prove is that

$$\operatorname{trace}((DA)^2)>0. \tag{4.3}$$

We have

$$\operatorname{trace}((DA)^2)=d_1^2a_{11}^2+2d_1d_2a_{12}a_{21}+d_2^2a_{22}^2.$$

If $a_{12}a_{21}\geqslant 0$ then, since $d_1^2a_{11}^2>0$ and $d_2^2a_{22}^2>0$, the inequality (4.3) holds. Note that this is also the case that A is sign symmetric and hence, by Theorem 4.2, the matrix $(DA)^2$ is a P-matrix. So, assume that $a_{12}a_{21}<0$. Since A^2 is a P_0^+-matrix, we have

$$(A^2)_{11}=a_{11}^2+a_{12}a_{21}\geqslant 0 \tag{4.4}$$

and

$$(A^2)_{22}=a_{12}a_{21}+a_{22}^2\geqslant 0, \tag{4.5}$$

with at least one of the two inequalities (4.4) and (4.5) strict. By multiplying (4.4) by d_1^2 and (4.5) by d_2^2 and adding the two resulting inequalities we obtain

$$d_1^2a_{11}^2+d_2^2a_{22}^2+a_{12}a_{21}(d_1^2+d_2^2)>0. \tag{4.6}$$

Since $d_1^2+d_2^2\geqslant 2d_1d_2$ and $a_{12}a_{21}<0$, it now follows from (4.6) that

$$0<d_1^2a_{11}^2+d_2^2a_{22}^2+(d_1^2+d_2^2)a_{12}a_{21}\leqslant d_1^2a_{11}^2+d_2^2a_{22}^2+2d_1d_2a_{12}a_{21},$$

proving (4.3). □

The two statements of Proposition 4.6 are equivalent in general for a certain class of matrices whose definition follows.

Definition 4.7. An $n\times n$ matrix A is said to be *anti-sign symmetric* if $A[\alpha|\beta]\times A[\beta|\alpha]\leqslant 0$ for all $\alpha,\beta\subset\{1,\dots,n\}$, $\alpha\neq\beta$, such that $|\alpha|=|\beta|$.

Theorem 4.8. *Let A be an anti-sign symmetric matrix. Then the following are equivalent*:

(i) *For every positive diagonal matrix D the matrix* $(DA)^2$ *is a Q-matrix.*
(ii) *The matrix* A^2 *is a* P_0^+*-matrix.*

Proof. (i) $\Rightarrow$ (ii) is proven in Proposition 4.4.

(ii) $\Rightarrow$ (i). Let A be an anti-sign symmetric $n \times n$ matrix such that the matrix A^2 is a P_0^+-matrix and let $D = \mathrm{diag}\{d_1, \ldots, d_n\}$ be a positive diagonal matrix. Since A^2 is a P_0^+-matrix and since D is a positive diagonal matrix it follows that:

$$(D[\alpha])^2 A^2[\alpha] \geqslant 0, \quad \forall \alpha \subset \{1, \ldots, n\}, \tag{4.7}$$

with strict inequality for at least one set α of each order k, $1 \leqslant k \leqslant n$. Using the Cauchy–Binet formula and summing over all subsets α of $\{1, \ldots, n\}$ of cardinality k, we obtain

$$\sum_{\substack{\alpha\subset\{1,\ldots,n\}\\|\alpha|=k}} (D[\alpha])^2 \sum_{\substack{\beta\subset\{1,\ldots,n\}\\|\beta|=|\alpha|}} A[\alpha|\beta]A[\beta|\alpha] > 0,$$

or

$$\sum_{\substack{\alpha,\beta\subset\{1,\ldots,n\}\\|\alpha|=|\beta|=k\\ \alpha\preceq\beta}} d_{\alpha\beta} A[\alpha|\beta]A[\beta|\alpha] > 0, \tag{4.8}$$

where "$\preceq$" denotes the lexicographic order relation, and where

$$d_{\alpha\beta} = \begin{cases} (D[\alpha])^2, & \alpha = \beta, \\ (D[\alpha])^2 + (D[\beta])^2, & \alpha \neq \beta. \end{cases}$$

In order to prove that $(DA)^2$ is a Q-matrix we have to show that for every positive integer k, $1 \leqslant k \leqslant n$ we have

$$\sum_{\substack{\alpha\subset\{1,\ldots,n\}\\|\alpha|=k}} (DA)^2[\alpha] > 0. \tag{4.9}$$

Using the Cauchy–Binet formula, we can write (4.9) as

$$\sum_{\substack{\alpha\subset\{1,\ldots,n\}\\|\alpha|=k}} \sum_{\substack{\beta\subset\{1,\ldots,n\}\\|\beta|=|\alpha|}} D[\alpha]D[\beta]A[\alpha|\beta]A[\beta|\alpha] > 0,$$

or

$$\sum_{\substack{\alpha,\beta\subset\{1,\ldots,n\}\\|\alpha|=|\beta|=k\\ \alpha\preceq\beta}} \hat{d}_{\alpha\beta} A[\alpha|\beta]A[\beta|\alpha] > 0, \tag{4.10}$$

where

$$\hat{d}_{\alpha\beta} = \begin{cases} (D[\alpha])^2, & \alpha = \beta, \\ 2D[\alpha]D[\beta], & \alpha \neq \beta. \end{cases}$$

Since $(D[\alpha])^2 + (D[\beta])^2 \geqslant 2D[\alpha]D[\beta] > 0$, and since by the anti-sign symmetry of A we have $A[\alpha|\beta]A[\beta|\alpha] \leqslant 0$ whenever $\alpha \neq \beta$, inequality (4.10) follows from inequality (4.8). □

As a corollary of Theorem 4.8 we obtain

Corollary 4.9. *Let A be an anti-sign symmetric P-matrix such that A^2 is a P_0^+-matrix. Then A is stable.*

Proof. The proof is essentially the same as the proof in [3] of Theorem 1.1. The only difference is that there the sign symmetry of a matrix A is used to prove that the square of every positive scaling DA is a P-matrix and thus has no negative eigenvalues, while here it follows by Theorem 4.8 that $(DA)^2$ is a Q-matrix and thus has no negative eigenvalues. □

5. On the relation between the spectra of P^S-matrices and Q^S-matrices

Since, by Theorem 1.3, P-matrices of order greater than 2 are not necessarily stable, and since in the proof of Theorem 1.1 the fact that A^2 is a P-matrix plays a crucial role, it is of interest to check a seemingly more restricted class, that is, matrices some of whose powers are P-matrices. In view of Theorem 1.2 it would be interesting to investigate also the relation between spectral properties of such matrices and matrices whose corresponding powers are Q-matrices. In this context, Hershkowitz and Johnson [10] posed the following question.

Question 5.1. Are the spectra of PM-matrices the same as those of QM-matrices?

It is shown in [10] that the answer to Question 5.1 is affirmative for matrices of order less than 5.

A more general version of Question 5.1 is the following.

Question 5.2. Let S be a (finite or infinite) set of positive integers. Are the spectra of P^S-matrices the same as those of Q^S-matrices?

In this section we shall answer this question affirmatively for matrices of order less than 4.

Definition 5.3

(i) A set Ω of complex numbers that serves as the spectrum of some P-matrix is said to be a P-set.

(ii) For a positive integer k we denote by Ω^k the set which consists of the kth powers of the elements of Ω. For a set S of positive integers, the set Ω is said to be a P^S-set if Ω^k is a P-set for all $k \in S$.

Remark 5.4. Let Ω be a set of complex numbers and let S be a set of positive integers. Note that Ω is a P^S-set if and only if it is the spectrum of some Q^S-matrix. Furthermore, every matrix with spectrum Ω is a Q^S-matrix.

In answering Question 5.2 for 2×2 matrices we shall use the following three lemmas.

Lemma 5.5. *A matrix A of the form*

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \tag{5.1}$$

is a P-matrix if and only if it is a Q-matrix.

Proof. The assertion follows since the diagonal elements of A are all equal, and so their sum is positive if and only if each one is positive. $\square$

It is easy to check that

Lemma 5.6. *A product of two matrices of the form* (5.1) *is of the same form.*

Finally,

Lemma 5.7. *For any complex number λ there exists a real matrix A of the form* (5.1) *with spectrum* $\{\lambda, \overline{\lambda}\}$.

Proof. The matrix is $\begin{bmatrix} \mathrm{Re}(\lambda) & \mathrm{Im}(\lambda) \\ -\mathrm{Im}(\lambda) & \mathrm{Re}(\lambda) \end{bmatrix}$. $\square$

We can now answer Question 5.2 for 2×2 matrices.

Theorem 5.8. *Let Ω be a set of two complex numbers and let S be a set of positive integers. The following are equivalent*:

(i) *The set Ω is the spectrum of some $2 \times 2 P^S$-matrix.*
(ii) *The set Ω is the spectrum of some 2×2 Q^S-matrix.*

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i). Let $\Omega = \{\lambda_1, \lambda_2\}$ be the spectrum of some 2×2 Q^S-matrix. Clearly, either $\lambda_1, \lambda_2 \in \mathbb{R}$ or $\lambda_1 = \overline{\lambda_2}$. If $\lambda_1, \lambda_2 \in \mathbb{R}$ then we choose the Q^S-matrix $A = \mathrm{diag}(\lambda_1, \lambda_2)$. Since, by Theorem 1.3, a real diagonal matrix is a Q-matrix if and only if it is a positive diagonal matrix and thus a P-matrix, it follows that A is a P^S-matrix with spectrum Ω. If $\lambda_1 = \overline{\lambda_2}$ then, by Lemma 5.7, there exists a matrix A of the form (5.1) with spectrum Ω. As is noted in Remark 5.4, since Ω is the spectrum of some

Q^S-matrix it follows that A is a Q^S-matrix. Since, by Lemma 5.6, all powers of A are of the form (5.1), it follows by Lemma 5.5 that A is a P^S-matrix. $\square$

We use a similar approach for 3×3 matrices.

Lemma 5.9. *A 3×3 circulant matrix A, that is, a matrix A of the form*

$$\begin{bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{bmatrix} \tag{5.2}$$

is a P-matrix if and only if it is a Q-matrix.

Proof. Since A has all equal diagonal elements and all equal 2×2 principal minors, it follows that the sums of principal minors of each order are positive if and only if every principal minor is positive. $\square$

Here too it is easy to check that

Lemma 5.10. *A product of two circulant matrices is again a circulant matrix.*

Finally,

Lemma 5.11. *For any real number r and a complex number λ there exists a real 3×3 circulant matrix A with spectrum $\{r, \lambda, \overline{\lambda}\}$.*

Proof. The matrix A of the form (5.2) for which

$$x = \frac{r + 2\mathrm{Re}(\lambda)}{3}, \quad y = \frac{r - \mathrm{Re}(\lambda) - \sqrt{3}\mathrm{Im}(\lambda)}{3},$$

$$z = \frac{r - \mathrm{Re}(\lambda) + \sqrt{3}\mathrm{Im}(\lambda)}{3}$$

is the required matrix. $\square$

The affirmative answer for Question 5.2 for 3×3 matrices follows:

Theorem 5.12. *Let Ω be a set of three complex numbers and let S be a set of positive integers. The following are equivalent:*

(i) *The set Ω is the spectrum of some 3×3 P^S-matrix.*
(ii) *The set Ω is the spectrum of some 3×3 Q^S-matrix.*

Proof. The proof is exactly the same as the proof of Theorem 5.8, where for the set $\Omega = \{\lambda_1, \lambda_2, \lambda_3\}$ we have either $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ or $\lambda_1 \in \mathbb{R}$ and $\lambda_2 = \overline{\lambda_3}$, where the form (5.2) replaces (5.1) and where Lemmas (5.9)–(5.11) replace Lemmas (5.5)–(5.7) correspondingly. □

It would be natural to try to generalize the above approach for real 4×4 circulant matrices, that is, real matrices of the form

$$\begin{bmatrix} x & y & z & w \\ w & x & y & z \\ z & w & x & y \\ y & z & w & x \end{bmatrix}. \tag{5.3}$$

Unfortunately, such a generalization does not hold. First, notice that in this case we do not have the analog of Lemmas 5.5 and 5.9 since not all 2×2 principal minors of a matrix of form (5.3) are equal. Furthermore, on one hand we have

Proposition 5.13. *All P-matrices of the form* (5.3) *are stable.*

Proof. The eigenvalues of a matrix A given by (5.3) are

$$x + y + z + w, \quad x - y + z - w, \quad x - z \pm \mathrm{i}(-y + w).$$

By Theorem 1.3, the two real eigenvalues of A are positive. Due to the positivity of the trace of A we have $x > 0$. Since all principal minors of A are positive and since $A[\{1, 3\}] = x^2 - z^2$, we have $x > z$. It now follows that the real parts of the complex eigenvalues are positive, and so the matrix is stable. □

On the other hand, in the sequel we shall show (Corollary 6.9) that for every finite set S of positive integers there exists an unstable P^S-set of cardinality 4. It thus follows that there exist P^S-sets Ω of cardinality 4 for which there exists no real matrix A of the form (5.3) with spectrum Ω.

6. Q^S-matrices and stability

Another question, which was suggested by Friedland (private communication) and formally posed in [10], is

Question 6.1. Let A be PM-matrix. Are all the eigenvalues of A positive real numbers?

It is shown in [10] that the answer to Question 6.1 is affirmative for matrices of order less than 5. Example 3.4 shows that for $n \geqslant 5$ QM-matrices are not necessarily stable. It thus follows that not both Questions 5.1 and 6.1 can have affirmative answers. In this section we refer to the relation between P^S-matrices or Q^S-matrices and stability, concentrating on P^2-matrices.

Motivated by Theorem 4.3, we pose the question what happens if we replace the requirement that "all positive scalings of A are P^2-matrices" by the weaker requirement that "A is a P^2-matrix". In particular, we ask

Question 6.2. Are P^2-matrices stable?

or even

Question 6.3. Are Q^2-matrices stable?

Remark 6.4. Another motivation to study Questions 6.2 and 6.3 is their relation to Question 6.1. Indeed, assume that there is an eigenvalue λ of a PM-matrix which is not a positive real number. Then there exists a power λ^n of λ which is not in the right half-plane. Note that λ^n is an eigenvalue of A^n which is a P^2-matrix, and thus we get a P^2-matrix which is not stable.

Clearly, a 1×1 Q-matrix has a positive eigenvalue. We answer Questions 6.2 and 6.3 positively also for 2×2 and 3×3 matrices using Theorem 1.3. By that theorem it immediately follows that for 2×2 we do not even need that A^2 is a Q-matrix.

Proposition 6.5. *Let A be a 2×2 Q-matrix. Then A is stable.*

Remark 6.6. It is easy to check that a set $\Omega = \{\lambda_1, \lambda_2\}$ is the spectrum of some $2 \times 2 Q^2$-matrix if and only if either $\lambda_1, \lambda_2 > 0$ or $\lambda_1 = \overline{\lambda_2}$ and $|\arg(\lambda)| < \pi/4$. By Theorem 5.8, such a set is also the spectrum of some P^2-matrix.

The stability of 3×3 Q^2-matrices is asserted in Proposition 3.1. The answer to Question 6.3 is negative for 4×4 matrices. In fact, we shall show that for every finite set S of positive integers there exists an unstable P^S-set of cardinality 4.

Lemma 6.7. *The set $\Omega = \{e^{(2\pi i/3)}, e^{-(2\pi i/3)}, a, b\}$, $a, b \in \mathbb{R}$, is a P-set if and only if $a + b > 1$ and either $0 < a, b < 1$ or $a, b > 1$ and $(a-1)(b-1) < 1$.*

Proof. The set Ω is a P-set if and only if all its elementary symmetric functions $\sigma_k(\Omega)$ are positive, that is

$$\sigma_1(\Omega) = -1 + a + b > 0, \tag{6.1}$$

$$\sigma_2(\Omega) = 1 + ab - a - b = (1-a)(1-b) > 0, \tag{6.2}$$

$$\sigma_3(\Omega) = a + b - ab = 1 - (a-1)(b-1) > 0, \tag{6.3}$$

and

$$\sigma_4(\Omega) = ab > 0. \tag{6.4}$$

Note that (6.1) and (6.4) hold if and only if $a + b > 1$ and $a, b > 0$. It thus follows that (6.2) holds if and only if either $a, b < 1$ or $a, b > 1$. Clearly, (6.3) holds if and only if

$$(a - 1)(b - 1) < 1. \tag{6.5}$$

Finally, note that if $0 < a, b < 1$ then (6.5) holds. □

As a consequence we obtain

Theorem 6.8. *Let S be a set of positive integers. The following are equivalent*:

(i) *There exists a* P^S*-set of the type* $\{e^{(2\pi i/3)}, e^{-(2\pi i/3)}, a, b\}$, $a, b \in \mathbb{R}$.
(ii) *The set S contains a finite number of integers that are not multiples of* 3.

Proof. Let S be a set of positive integers, let S_1 be the set of all elements of S which are multiples of 3, let $S_2 = S \setminus S_1$, and let $\Omega = \{e^{(2\pi i/3)}, e^{-(2\pi i/3)}, a, b\}$. The proof of the equivalence follows.

(i) ⇒ (ii). Note that for every $k \in S_2$ we have $\Omega^k = \{e^{(2\pi i/3)}, e^{-(2\pi i/3)}, a^k, b^k\}$. If S_2 is infinite then, since if $0 < a, b < 1$ then for $k \in S_2$ sufficiently large we have $a^k + b^k \leqslant 1$ and if $a, b > 1$ then for $k \in S_2$ sufficiently large we have $(a^k - 1)(b^k - 1) \geqslant 1$, it follows by Lemma 6.7 that Ω is not a P^{S_2}-set and thus not a P^S-set.

(ii) ⇒ (i). Assume that S_2 is finite. Note that for every $k \in S_1$ we have $\Omega^k = \{1, 1, a^k, b^k\}$, which is a P-set if and only if $a^k, b^k > 0$. Therefore, if $a, b > 0$ then Ω is a P^{S_1}-set. Now, let n be the largest element of the finite set S_2, and choose a and b such that $\sqrt[n]{1/2} < a, b < 1$. Since $0 < a^k, b^k < 1$ and $a^k + b^k > 1$ whenever $k \leqslant n$, it follows from Lemma 6.7 that Ω is a P^{S_2}-set. Since Ω is also P^{S_1}-set, it follows that Ω is a P^S-set. □

Since $e^{\pm(2\pi i/3)}$ has a negative real part, it now follows that:

Corollary 6.9. *Let S be a finite set of positive integers. There exists an unstable* P^S*-set of cardinality* 4.

Theorem 6.8 can be used to provide an alternative elementary affirmative answer for Questions 5.1 and 6.1 for 4×4 matrices, originally answered in [10]. In our proof we use the following lemma, which is also used in [10].

Lemma 6.10 [10, Lemma 1]. *Let* λ *be an eigenvalue of an* $n \times n$ QM*-matrix. Then* $\arg(\lambda)$ *is a rational multiple of* 2π. *Furthermore, if this multiple has denominator d in reduced form then d is odd and* $d < n$.

Theorem 6.11. *Let S be a* PM*-set of order* 4. *Then all the elements of S are positive real numbers.*

Proof. By Lemma 6.10, the elements of S can be positive numbers or complex numbers of the form $r\mathrm{e}^{\pm(2\pi\mathrm{i}/3)}$. If all four elements of S are of the latter type then they all have negative real parts, implying that their sum is negative and so S is not even a P-set. By Theorem 6.8 the set S cannot be of the form $\{r\mathrm{e}^{(2\pi\mathrm{i}/3)}, r\mathrm{e}^{-(2\pi\mathrm{i}/3)}, a, b\}$, $a, b \in \mathbb{R}$. Our claim follows. $\square$

Motivated by Corollary 6.9 and Theorem 6.11, we conclude our paper with the following open problem.

Problem 6.12. While by Theorem 6.11, 4×4 QM-matrices are stable, Corollary 6.9 asserts that for any finite set S of positive integers, 4×4 Q^S-matrices are not necessarily stable. Thus, one can ask

For which infinite sequences S of positive integers, all 4×4 Q^S*-matrices are stable*?

Note that Example 3.4 shows that for $n > 4$ even a sign symmetric QM-matrix is not necessarily stable. For a related discussion see [7,11].

References

[1] V.A. Babenko, N.M. Petrov, The *P*-matrix approach for the potential description of hadron–hadron interaction, Ukrain. Fiz. Zh. 32 (1987) 971–977.
[2] C.S. Ballantine, Stabilization by a diagonal matrix, Proc. Amer. Math. Soc. 25 (1970) 728–734.
[3] D. Carlson, A class of positive stable matrices, J. Res. Nat. Bur. Standards 78B (1974) 1–2.
[4] G.W. Cross, Three types of matrix stability, Linear Algebra Appl. 20 (1978) 253–263.
[5] M. Fiedler, V. Ptak, On matrices with non-positive off-diagonal elements and positive principal minors, Czechoslovak Math. J. 12 (1962) 382–400.
[6] F.R. Gantmacher, The Theory of Matrices, Chelsea, NY, 1959.
[7] G. Hanoch, D. Hershkowitz, Forcing sequences of positive integers, Czechoslovak Math. J. 45 (1995) 149–169.
[8] D. Hershkowitz, Stable matrices and matrices with nonnegative principal minors, Ph.D. thesis, Technion-Israel Institute of Technology, 1982.
[9] D. Hershkowitz, Recent directions in matrix stability, Linear Algebra Appl. 171 (1992) 161–186.
[10] D. Hershkowitz, C.R. Johnson, Spectra of matrices with *P*-matrix powers, Linear Algebra Appl. 80 (1986) 159–171.
[11] D. Hershkowitz, H. Schneider, Sequences, wedges, and associated sets of complex numbers, Czechoslovak Math. J. 38 (1986) 138–156.
[12] O. Holtz, Not all GKK τ-matrices are stable, Linear Algebra Appl. 291 (1999) 235–244.
[13] R.B. Kellogg, On complex eigenvalues of *M*- and *P*-matrices, Numer. Math. 19 (1972) 170–175.
[14] D. Koteljanskij, A property of sign symmetric matrices, Amer. Math. Soc. Transl. Ser. 2 27 (1963) 19–23.
[15] L. Metzler, Stability of multiple markets: the Hick conditions, Econometrica 13 (1945) 277–292.
[16] T. Parthasarthy, Application of *P*-matrices to economics, Applied Abstract Algebra, University of New Delhi, 1996, pp. 78–95.
[17] O. Taussky, Research problem, Bull. Amer. Math. Soc. 64 (1958) 124.